

Gauge transformations are canonical transformations, redux

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In this short note we return to the old paper by Tai L. Chow (Eur. J. Phys. **18**, 467–468 (1997)) and correct its erroneous final part. We also note that the main result of that paper, that gauge transformations of mechanics are canonical transformations, was known much earlier.

INTRODUCTION

The letter of Tai L. Chow [1] generated quite a wake in the literature [2–5], partly because its final part is erroneous, unfortunately. These errors, as well as other parts of the paper [1], are reproduced in Portuguese in [2] (by the other author), and they became a subject of consideration in two eprints [3] and [4]. Nevertheless, we feel some comments are still necessary as the error was not fixed. It is the aim of this short note to correct the proof of [1] that gauge transformations of mechanics are canonical transformations.

GAUGE TRANSFORMATIONS AS CANONICAL TRANSFORMATIONS

It is well known that for a dynamical system two Lagrangians are equivalent if they differ by a total time-derivative of any function of generalized coordinates and time:

$$L'(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt}f(q, t). \quad (1)$$

The relation (1) generates the following transformation of generalized coordinates and momenta:

$$\begin{aligned} Q_i &= q_i, \\ P_i &= \frac{\partial L'}{\partial \dot{q}_i} = p_i + \frac{\partial f(q, t)}{\partial \dot{q}_i}, \end{aligned} \quad (2)$$

because

$$\frac{d}{dt}f(q, t) = \sum_k \dot{q}_k \frac{\partial f(q, t)}{\partial q_k} + \frac{\partial f(q, t)}{\partial t}. \quad (3)$$

It is claimed in [1] that the transformation (2) is a canonical transformation, that is it preserves the Hamilton's equations:

$$\begin{aligned} \dot{Q}_i &= \frac{\partial H'(Q, P, t)}{\partial P_i}, \\ \dot{P}_i &= -\frac{\partial H'(Q, P, t)}{\partial Q_i}, \end{aligned} \quad (4)$$

where the new Hamiltonian $H'(Q, P, t)$ is related to the old one, $H(q, p, t)$, in the following way

$$H'(Q, P, t) = \sum_i P_i \dot{Q}_i - L' = H(Q, p, t) - \frac{\partial f(Q, t)}{\partial t}, \quad (5)$$

and here, according to (2),

$$p_i = P_i - \frac{\partial f(Q, t)}{\partial Q_i}. \quad (6)$$

It should be noted that [1] was not the first paper where it was stated that the transformation (2), induced by (1), is a canonical transformation. This fact was known long ago [6] and several proofs of it can be envisaged.

For example, one can explicitly construct the generating function [6]

$$\Phi(q, P, t) = \sum_i q_i P_i - f(q, t), \quad (7)$$

so that

$$p_i = \frac{\partial \Phi(q, P, t)}{\partial q_i}, \quad Q_i = \frac{\partial \Phi(q, P, t)}{\partial P_i}, \quad H' = H + \frac{\partial \Phi(q, P, t)}{\partial t}. \quad (8)$$

Just this function, and not the function f , as erroneously claimed in [1], is the generating function of the canonical transformation (2). What this transformation is the canonical transformation follows then from

$$\det \left(\frac{\partial^2 \Phi}{\partial q_i \partial P_j} \right) \neq 0. \quad (9)$$

Another standard way to prove that (2) is a canonical transformation is to calculate the fundamental Poisson brackets [3]:

$$\begin{aligned} \{Q_i, Q_j\} &= \sum_k \left(\frac{\partial Q_i}{\partial p_k} \frac{\partial Q_j}{\partial q_k} - \frac{\partial Q_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} \right) = 0, \\ \{P_i, P_j\} &= \sum_k \left(\frac{\partial P_i}{\partial p_k} \frac{\partial P_j}{\partial q_k} - \frac{\partial P_i}{\partial q_k} \frac{\partial P_j}{\partial p_k} \right) = 0, \\ \{P_i, Q_j\} &= \sum_k \left(\frac{\partial P_i}{\partial p_k} \frac{\partial Q_j}{\partial q_k} - \frac{\partial P_i}{\partial q_k} \frac{\partial Q_j}{\partial p_k} \right) = \delta_{ij}. \end{aligned} \quad (10)$$

Still another way, that was chosen in [1], is to directly verify the validity of the new Hamilton equations (4). Unfortunately, by some mysterious reason, it escaped the attention of both of the author and of the referee that the form of Hamilton's equations used in the final part of [1] was erroneous thus invalidating the otherwise correct conclusions of [1]. The error was noticed in [3] and [4]. Reference [3], as was mentioned above, gave a different proof of the main claim of [1], based on fundamental Poisson brackets, but has not corrected the treatment of [1]. Reference [4], on the contrary, had tried to correct the error, but, unfortunately, introducing its own mistakes, it came to the wrong conclusion that the main result of [1] was incorrect.

In fact, the validity of the first equation of (4) is not difficult to prove:

$$\frac{\partial H'(Q, P, t)}{\partial P_i} = \sum_k \frac{\partial H(Q, p, t)}{\partial p_k} \frac{\partial p_k}{\partial P_i} = \frac{\partial H(Q, p, t)}{\partial p_i} = \dot{Q}_i. \quad (11)$$

It is the second equation of (4) the proof of which contains some subtleties causing an error in [4]. The subtlety is that while calculating the partial derivative $\frac{\partial H'(Q, P, t)}{\partial Q_i}$ it is the new momentum P and not the old one p which is held fixed. In other words, while calculating this partial derivative, we should take into account that p , according to (6), is also a function of Q . Then we have

$$\begin{aligned} \frac{\partial H'(Q, P, t)}{\partial Q_i} &= \frac{\partial H(Q, p, t)}{\partial Q_i} + \sum_k \frac{\partial H(Q, p, t)}{\partial p_k} \frac{\partial p_k(Q, P, t)}{\partial Q_i} - \\ \frac{\partial^2 f(Q, t)}{\partial Q_i \partial t} &= -\dot{p}_i - \sum_k \dot{Q}_k \frac{\partial^2 f(Q, t)}{\partial Q_k \partial Q_i} - \frac{\partial^2 f(Q, t)}{\partial Q_i \partial t}. \end{aligned} \quad (12)$$

But

$$\sum_k \dot{Q}_k \frac{\partial^2 f(Q, t)}{\partial Q_k \partial Q_i} + \frac{\partial^2 f(Q, t)}{\partial Q_i \partial t} = \frac{d}{dt} \frac{\partial f(Q, t)}{\partial Q_i}, \quad (13)$$

and we get finally

$$\frac{\partial H'(Q, P, t)}{\partial Q_i} = -\frac{d}{dt} \left[p_i + \frac{\partial f(Q, t)}{\partial Q_i} \right] = -\dot{P}_i. \quad (14)$$

ELECTROMAGNETIC GAUGE TRANSFORMATIONS

Interestingly, electromagnetic gauge transformations

$$\vec{A}' = \vec{A} + \nabla\Lambda(\vec{r}, t), \quad \phi' = \phi - \frac{\partial\Lambda(\vec{r}, t)}{\partial t} \quad (15)$$

induce the transformation of the type (1) in the Lagrangian of a classical charged particle in an electromagnetic field [7]. Indeed, if we substitute $\vec{A} \rightarrow \vec{A}'$, $\phi \rightarrow \phi'$ from (15) into the Lagrangian

$$L = \frac{1}{2}m\dot{\vec{r}}^2 - e\phi(\vec{r}, t) + e\dot{\vec{r}} \cdot \vec{A}, \quad (16)$$

we get a new Lagrangian in the form

$$L' = L + e\frac{\partial\Lambda(\vec{r}, t)}{\partial t} + e\dot{\vec{r}} \cdot \nabla\Lambda(\vec{r}, t) = L + e\frac{d\Lambda(\vec{r}, t)}{dt}. \quad (17)$$

As we see, the resulting transformation of the Lagrangian is of the form (1) with $f = e\Lambda$ [7].

That electromagnetic gauge transformations induce canonical transformations in the charged particle Lagrangian is well known [6–8]. We see that [1], when corrected, gives the proof of this classic result too.

CONCLUDING REMARKS

That gauge transformations are a subset of canonical transformations is, of course, known for a long time. Nevertheless, perhaps it is worthwhile to correct the proof of [1] because it generated some confusion in the literature. We have provided the necessary amendments in this note. Besides we have indicated two other methods of proof. So we hope the case is settled now: “I have said it thrice: What I tell you three times is true” [9].

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